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Geometrical Derivation of the Intrinsic Fokker–Planck Equation and Its Stationary Distribution

J. Masoliver,[1,2] L. Garrido,[1] and J. Llosa[1]

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Starting from an intrinsic Langevin equation, we give a geometrical derivation of the Fokker–Planck equation. We also present a method for obtaining a stationary distribution and for deriving potential conditions when the diffusion matrix is singular.

KEY WORDS: Intrinsic Langevin and Fokker–Planck equations; stationary distributions; potential conditions; singular diffusion matrix; invariants and symmetries.

1. INTRODUCTION

For many years efforts have been devoted to finding a covariant formulation of the Fokker–Planck equation (FPE) and to applying it to physical systems.[(1–12)] Clearly, the principle of covariance affirms that all physical laws can be written by means of equations that maintain their form with respect to general transformations of coordinates.

Another equivalent, although more powerful formulation than the covariant one is the intrinsic formulation. An equation is intrinsic when it is completely independent of the coordinate systems, so any property that may be deduced from an intrinsic equation represents the physical reality in any coordinate system. Recently, an intrinsic formulation of the Fokker–Planck equation has been obtained.[(13)]

Our aim in this paper is to perform a geometrical (i.e., intrinsic)

[1] Departament de Fisica Tèorica, 08028 Barcelona, Universitat de Barcelona, Spain.
[2] Present Address: Institute for Non-Linear Science, University of California—San Diego, La Jolla, California 92093.

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derivation of the Fokker–Planck equation from the Langevin equation. In this way we arrive at a FPE written in a different intrinsic notation than that used in Ref. 13: it has the additional advantage that one need not select a metric tensor of phase space.[3] This intrinsic Fokker–Planck equation also gives a systematic method of finding the stationary distribution when detailed balance is present but the diffusion matrix is singular.[14]

In Section 2 we give the geometrical derivation of the FPE starting from the Langevin equation and using functional techniques.[15,16] In Section 3 we find the stationary distribution generalizing the potential conditions for those cases in which the diffusion matrix is singular and we apply the method to practical examples of physical interest.

2. GEOMETRICAL DERIVATION OF THE FOKKER–PLANCK EQUATION

We begin with the Langevin equation:

$$\dot{q}^\mu = f^\mu(\mathbf{q}) + g_k^\mu(\mathbf{q}) \cdot \xi^k(t) \tag{2.1}$$

(with implied summation over repeated indices). This equation gives the temporal evolution of the trajectories $q^\mu(t)$, $\mu = 1,..., n$, of the ensemble of gross variables. The functions $\xi^k(t)$, $k = 1,..., m$, are fluctuating functions in time and their statistical properties are known. For certain initial conditions, $q_0^\mu \equiv q^\mu(0)$. The solution of (2.1), $\mathbf{q}(t, \xi)$, is a well-defined stochastic processes, and for each realization of $\xi(t)$ the solution of (2.1) yields a well-defined trajectory in phase space.[10,17,18] The Langevin equation is a covariant equation, since under a general change of coordinates the magnitudes $\dot{q}^\mu$, f^μ, g_k^μ transform into contravariant vectors.[3,4] Now our first step is to write Eq. (2.1) in an intrinsic form.

Let M be the phase space of the system that has the structure of a differentiable manifold of n dimensions, and let $T(M)$ be the space of the vector fields tangent to the manifold M.[19] From the Langevin equation (2.1) we define the following fields of vectors tangent to M:

$$F \equiv f^\mu(\mathbf{q}) \cdot \partial/\partial q^\mu, \qquad F \in T(M) \tag{2.2a}$$

$$G_k \equiv g_k^\mu(\mathbf{q}) \cdot \partial/\partial q^\mu, \qquad G_k \in T(M) \tag{2.2b}$$

$$X_0 \equiv \dot{q}^\mu \cdot \partial/\partial q^\mu, \qquad X_0 \in T(M) \tag{2.2c}$$

[3] The selection of metric tensor is not unique. Usually the diffusion matrix is chosen as the metric tensor of the phase space.[1,2,4,5,7–9,13] This choice restricts the formalism to nonsingular diffusion matrices. This does not happen here, where that selection is not necessary.

These vector fields do not depend on the system of gross variables (they are intrinsic). Introducing Eqs. (2.2) into Eq. (2.1), we have

$$X_0 = F + G_k \xi^k(t), \qquad k = 1, \ldots, m \tag{2.3}$$

Defining the tangent field $X \in T(R \times M)$ by

$$X \equiv \partial/\partial t + \dot{q}^\mu \, \partial/\partial q^\mu \tag{2.4}$$

we find that the Langevin equation (2.3) becomes

$$X = \partial/\partial t + F + G_k \xi^k(t), \qquad k = 1, \ldots, m \tag{2.5}$$

which is completely independent of the chosen system of gross variables.

As is well known, if the noise functions $\xi^k(t)$ are Gaussian and delta-correlated (white noise), the Langevin equation is stochastically equivalent to the Fokker–Planck equation for the probability density.[4] Therefore, our next step will be to carry out a geometrical derivation of the FPE starting from the intrinsic Langevin equation with white noise and using functional methods.

Let[5] $\phi^\mu(t; t_0, \mathbf{q}_0; [\xi])$ be the solution of Eq. (2.5) with the deterministic initial conditions

$$\phi^\mu(t_0; t_0, q_0; [\xi]) = q_0^\mu \tag{2.6}$$

The integral curves of the Langevin equation generate a transformation $R \times M$ that we can write in the form

$$\begin{aligned} \psi_\tau: \quad & R \times M \to R \times M \\ & (t_0, \mathbf{q}_0) \to \psi_\tau(t_0; \mathbf{q}_0; [\xi]) \end{aligned} \tag{2.7}$$

where $\psi_\tau(t_0, \mathbf{q}_0; [\xi])$ is the point

$$\psi_\tau(t_0; q_0; [\xi]) \equiv (\tau + t_0, \boldsymbol{\phi}(\tau; t_0, \mathbf{q}_0; [\xi]))$$

in the extended phase space $R \times M$.

If $\Lambda(R \times M)$ is the space of the forms defined in $R \times M$,[19] the reciprocal image ψ_τ^* associated with (2.6) is the mapping that turns forms defined in the point $\psi_\tau(t_0, \mathbf{q}_0)$ into forms defined in the point $(t_0, \mathbf{q}_0)$,[19]

$$\Lambda(R \times M)_{(t_0, \mathbf{q}_0)} \underset{\psi_\tau^*}{\overset{\psi^*_{-\tau}}{\rightleftarrows}} \Lambda(R \times M)_{\psi_\tau(t_0, \mathbf{q}_0)}$$

[4] In what follows we will assume the Stratonovich interpretation rule[3,29] for the stochastic integrals.

[5] By $[\xi]$ we mean the functional dependences on the noise $\xi^k(t)$.

Let

$$\rho(\mathbf{q}, t) \equiv \delta^n(\mathbf{q} - \boldsymbol{\phi}(t; t_0, \mathbf{q}_0; [\xi])) \tag{2.8}$$

be the density of points belonging to the trajectories of the Langevin equation. We define the *n-form number of points*

$$N \equiv \rho(q, t)\, dq^1 \wedge \cdots \wedge dq^n \in \Lambda^n(R \times M) \tag{2.9}$$

(the symbol $\wedge$ denotes the exterior product[20]). The $(n+1)$-form

$$\Omega \equiv N \wedge dt \tag{2.10}$$

represents the number of points located between $\mathbf{q}$ and $\mathbf{q} + d\mathbf{q}$ during the time interval $(t, t+dt)$. After a certain time interval τ, these points will evolve, following (2.6), toward $\psi_\tau(t, \mathbf{q})$. Then the form $\Omega_{(t,\mathbf{q})}$ becomes $\psi^*_{-\tau}[\Omega_{(t,\mathbf{q})}]$. Assuming the conservation of the number of points, we have

$$\psi^*_{-\tau}[\Omega_{(t,\mathbf{q})}] = \Omega_{\psi_\tau(t,\mathbf{q})}$$

that is,

$$\lim_{\tau \to 0} \frac{\psi^*_{-\tau}[\Omega_{(t,\mathbf{q})}] - \Omega_{\psi_\tau(t,\mathbf{q})}}{\tau} = 0$$

The left-hand side of this equation is the Lie derivative of the form Ω in the direction of the vector field X,[19] so we have the conservation equation

$$L_X \Omega = 0 \tag{2.11}$$

which, taking account the linearity of the Lie derivative, can be written in the form [see Eq. (2.5)]:

$$L_{\partial/\partial t}\Omega + L_F \Omega + L_{\xi^k G_k}\Omega = 0 \tag{2.12}$$

However (see Appendix A),

$$L_{\xi^k G_k}\Omega = L_{G_k}(\xi^k \Omega) \tag{2.13}$$

Introducing (2.13) into (2.12) and averaging over all noises $\xi^k(t)$, we get

$$L_{\partial/\partial t}\langle \Omega \rangle + L_F \langle \Omega \rangle + L_{G_k}\langle \xi^k \Omega \rangle = 0 \tag{2.14}$$

where

$$\langle \Omega \rangle \equiv \langle \delta(\mathbf{q} - \boldsymbol{\phi}(t; t_0, \mathbf{q}_0; [\xi])) \rangle\, dq^1 \wedge \cdots \wedge dq^n \wedge dt \tag{2.15}$$

is the $(n+1)$-*form of probability*.

For Gaussian white noise we show in Appendix A that

$$\langle \xi^k(t)\, \Omega \rangle = -\frac{1}{2}\, \delta^{ik}\, L_{G_j} \langle \Omega \rangle \tag{2.16}$$

where δ^{jk} is the Kronecker symbol. Substitution of Eq. (2.16) in Eq. (2.14) yields

$$L_{\partial/\partial t} \langle \Omega \rangle + L_F \langle \Omega \rangle - \frac{1}{2}\, \delta^{kj}\, L_{G_k} L_{G_j} \langle \Omega \rangle = 0 \tag{2.17}$$

Finally, defining the *n-form of probability*

$$\Pi \equiv \langle N \rangle = \langle \rho(\mathbf{q}, t) \rangle\, dq^1 \wedge \cdots \wedge dq^n \equiv P(\mathbf{q}, t)\, dq^1 \wedge \cdots \wedge dq^n \tag{2.18}$$

we can write Eq. (2.17) as (see Appendix A)

$$\dot{\Pi} = -L_F \Pi + \frac{1}{2}\, \delta^{jk}\, L_{G_j} L_{G_k} \Pi \tag{2.19}$$

which is the *intrinsic Fokker–Planck equation* (in the sense of Stratonovich).[3] Note that to arrive to this equation it has not been necessary to define any metric tensor in the manifold M.[6]

By means of the relation

$$L_X = d i_X + i_X d \tag{2.20}$$

where d is the exterior derivative and i_X is the interior product associated to the field X,[19,21] and by taking into account that

$$d\Pi = 0 \tag{2.21}$$

(since Π is an n-form or an n-manifold $M^{(20)}$), we can write Eq. (2.19) as a continuity equation:

$$\dot{\Pi} + d\Gamma = 0 \tag{2.22}$$

where the $(n-1)$-form given by

$$\Gamma \equiv i_F \Pi - \frac{1}{2}\, \delta^{jk}\, i_{G_j}(L_{G_k} \Pi) \tag{2.23}$$

$[\Gamma \in \Lambda^{n-1}(M)]$ is the *probability current*.

[6] This agrees with the fact that the FPE does not depend on any metric structure of the phase space, since the FPE may be viewed as a continuity equation which involves only the definition of the volume element of the phase space.

3. STATIONARY DISTRIBUTION

For the stationary case we have

$$\dot{\Pi}_{\text{st}} = 0 \tag{3.1}$$

which, when introduced in Eq. (2.22), implies

$$d\Gamma_{\text{st}} = 0 \tag{3.2}$$

A particular solution is that corresponding to vanishing probability current *(the potential case)*[14,22]:

$$\Gamma_{\text{st}} = i_F \Pi_{\text{st}} - \frac{1}{2}\delta^{jk}\, i_{G_j} L_{G_k} \Pi_{\text{st}} = 0 \tag{3.3}$$

As is well known, if the drift and the diffusion matrix (which has to be nonsingular) satisfy certain relations called *potential conditions*, then the probability current vanishes for all points in the phase space and the stationary distribution may be written in a closed form.[14,22] We will study that case by using the formalism presented here. First, we must find the conditions under which Eq. (3.3) has nontrivial solutions.

Since $L_{G_k}\Pi_{\text{st}}$ is an n-form, it can be expressed as

$$L_{G_k}\Pi_{\text{st}} = \lambda_k(\mathbf{q})\, \Pi_{\text{st}}, \qquad k = 1,\dots, m \tag{3.4}$$

where $\lambda_k(\mathbf{q})$ are k functions on the phase space. Introducing (3.4) into Eq. (3.3), we arrive, after some manipulations, at

$$i_H \Pi_{\text{st}} = 0 \tag{3.5}$$

where the field H is given by

$$H \equiv F - \frac{1}{2}\sum_{j=1}^{m} \lambda_j(\mathbf{q})\, G_j \tag{3.6}$$

Equation (3.5) implies

$$H \equiv F - \frac{1}{2}\sum_{j=1}^{m} \lambda_j(\mathbf{q})\, G_j = 0 \tag{3.7}$$

Therefore, the following are the conditions for Eq. (3.3) to have a solution:

(i) The vector field F must depend linearly on $G_1,\dots, G_m$; i.e., there must be m functions $\lambda_1(\mathbf{q}),\dots, \lambda_m(\mathbf{q})$ such that:

$$F = \frac{1}{2}\sum_{j=1}^{m} \lambda_j(\mathbf{q})\, G_j \tag{3.8}$$

(ii) The solution Π_{st} of Eq. (3.3) must then satisfy the m (partial differential) equations

$$L_{G_k}\Pi_{st} = \lambda_k(\mathbf{q})\,\Pi_{st}, \qquad k=1,..., m \tag{3.9}$$

where the $\lambda(\mathbf{q})$ are precisely the coefficients on the right-hand side of Eq. (3.8). In every system of gross variables Eq. (3.9) reads [cf. (2.18) and Appendix A]

$$g_k^\mu(\mathbf{q})\frac{\partial P_{st}(\mathbf{q})}{\partial q^\mu} + \left[\frac{\partial g_k^\mu(\mathbf{q})}{\partial q^\mu} - \lambda_k(\mathbf{q})\right] P_{st}(\mathbf{q}) = 0, \qquad k=1,..., m \tag{3.10}$$

Equation (3.9) or (3.10) is a first-order partial differential system, the integrability conditions of which turn out to be a straight consequence of the Frobenius theorem. Thus the system (3.9) [or (3.10)] is integrable if it is complete, i.e., if the set of vector fields[7] $\{G_\alpha, \alpha=1,..., m\}$ is closed under the Lie bracket[(19,23)]

$$[G_\alpha, G_\beta] = C_{\alpha\beta}^\gamma(\mathbf{q})\,G_\gamma, \qquad \alpha,\beta,\gamma=1,..., m \tag{3.11}$$

where $[G_\alpha, G_\beta] \equiv G_\alpha G_\beta - G_\beta G_\alpha$ is the Lie bracket and $C_{\alpha\beta}^\gamma(\mathbf{q})$ are real functions. If the set $\{G_\alpha, \alpha=1,..., m\}$ is not closed under the Lie bracket, we have to extend the set by those Lie brackets $[G_i, G_j]$, $i, j=1,..., m$, that do not depend linearly on $G_1,..., G_m$. This extension is shown in Appendix B.

In what follows we assume that the system

$$L_{G_\alpha}\Pi_{st} = \lambda_\alpha \Pi_{st}, \qquad \alpha=1,..., m \tag{3.12}$$

is complete (or that it has been completed). The integrability conditions of (3.12) are those derived by applying the identity[(23)]

$$[L_{G_\alpha}, L_{G_\beta}] = L_{[G_0, G_\beta]} \tag{3.13}$$

to the system itself. Using Eq. (3.11), we obtain the integrability conditions for (3.12) as

$$L_{G_{\alpha\beta}^\gamma G_\gamma} \quad \Pi_{st} = (G_\alpha\lambda_\beta - G_\beta\lambda_\alpha)\,\Pi_{st} \tag{3.14}$$

which in every system of gross variables is

$$g_\gamma^\mu(\mathbf{q})\frac{\partial C_{\alpha\beta}^\gamma(\mathbf{q})}{\partial q^\mu} + C_{\alpha\beta}^\gamma(\mathbf{q})\,\lambda_\gamma(\mathbf{q}) - g_\alpha^\mu(\mathbf{q})\frac{\partial\lambda_\beta(\mathbf{q})}{\partial q^\mu}$$
$$+\; g_\beta^\mu(\mathbf{q})\frac{\partial\lambda_\alpha(\mathbf{q})}{\partial q^\mu} = 0, \qquad \alpha,\beta=1,..., m \tag{3.15}$$

where the $C_{\alpha\beta}^\gamma(\mathbf{q})$ are given by Eq. (3.11).

[7] We assume that the noise vector fields G_α are linearly independent.

Therefore, if the vector fields F and G_α satisfy the conditions given by Eqs. (3.8) and (3.14) [or (3.15)], then we have zero probability current and the stationary distribution is the solution of Eq. (3.9) [or (3.10)].

In the special case $m = n$ (equal number of independent noise functions and gross variables, which implies a nonsingular diffusion matrix), the condition (3.8) is automatically fulfilled and the system (3.9) is itself complete. In this case the so-called potential conditions can be derived, after some manipulations, from Eq. (3.15). Thus, we may view Eq. (3.15) as a generalization of the potential conditions when the diffusion matrix is singular. For $m = 1$ (a unique noise function), Eqs. (3.8) and (3.9) become, respectively,

$$F = \frac{1}{2} \lambda(\mathbf{q})\, G \tag{3.16}$$

and

$$L_G \Pi_{\text{st}} = \lambda(\mathbf{q})\, \Pi_{\text{st}} \tag{3.17}$$

the latter being a first-order partial differential equation, which is always integrable.

As an example, we will apply the formalism hitherto studied to two-dimensional Langevin equations, which are relevant in the study of non-linear optics.

A simple model to study optical bistability[24] consists of an ensemble of homogeneous two-level atoms interacting with an electromagnetic field. This system is described by coupled Maxwell–Bloch equations in the rotating-wave approximation. These equations involve the internal electric field amplitude, the polarization of the medium, and the inversion density.[25] Neglecting the spatial variation of the field amplitudes and assuming that the atomic response to the electromagnetic field is very fast, one can make the adiabatic approximation,[25] which leads to a single first-order equation for the electromagnetic field. When the frequencies of the empty resonator, the two-level system, and the driving field are very similar and the exterior pumping term is negligible, the equation for the electromagnetic field is[26,27]

$$\dot{E}^+ = -E^+ \left(1 + \Gamma^2 \frac{1}{1 + |E|^2}\right) + F^+(t) + \frac{E^+}{1 + |E|^2}\, \Gamma(t) \tag{3.18}$$

where E^+ is the complex field amplitude and Γ^2 is a real parameter. $F^+(t)$ and $\Gamma(t)$ are fluctuating forces that in first approximation are assumed to be white Gaussian noise; the fluctuations of the polarization and of the field are included in $F^+(t)$. The $\Gamma(t)$ takes account of the inversion fluctuations.

Following Schenzle and Brand,[26] if we neglect the fluctuation of the polarization and of the field, we obtain a two-dimensional Langevin equation with pure multiplicative fluctuations:

$$\dot{E}^+ = -E^+\left(1+\Gamma^2\frac{1}{1+|E|^2}\right)+\frac{E^+}{1+|E|^2}\Gamma(t) \tag{3.19}$$

In fact, Schenzle and Brand go one step further, assuming that the field intensity $|E|^2$ is small and expanding the saturation term up to first order in the field intensity. After neglecting higher order terms in the fluctuations, the final result is

$$\dot{E}^+ = -(1+\Gamma^2)\,E^+ + \Gamma^2\,|E|^2\,E^+ + E^+\Gamma(t) \tag{3.20}$$

This latter equation is a model for the laser transition with pure multiplicative inversion fluctuations. If we assume that the inversion fluctuations $\Gamma(t)$ are real, we can write Eqs. (3.19) and (3.20) in a slightly more general form:

$$\dot{z} = -\left(a+\frac{b}{1+|z|^2}\right)z+\frac{z}{1+|z|^2}\xi(t) \tag{3.21}$$

and

$$\dot{z} = (\alpha-\beta\,|z|^{2\gamma})\,z + z\xi(t) \tag{3.22}$$

where z represents the complex field amplitude, $a, b, \alpha, \beta, \gamma \geqslant 0$ are real parameters, and $\xi(t)$ is a real, Gaussian white noise.[8]

Putting $z = x_1 + ix_2$, we find that Eqs. (3.21) and (3.22) are equivalent to the equations

$$\dot{x}_\mu = \left(a+\frac{b}{1+x_1^2+x_2^2}\right)x_\mu+\frac{x_\mu}{1+x_1^2+x_2^2}\,\xi(t), \qquad \mu=1,2 \tag{3.23}$$

and

$$\dot{x}_\mu = \alpha x_\mu - \beta(x_1^2+x_2^2)^\gamma\,x_\mu + x_\mu\xi(t), \qquad \mu=1,2 \tag{3.24}$$

In both cases there are fewer noise functions than gross variables.[9,18] This yields singular diffusion matrices and the notmal procedure for finding

[8] Equation (3.22) also represents a suitable model for subharmonic generation, parametric three-wave mixing, and Raman scattering, as well as autocatalytic reactions.[26]

stationary distributions, via potential conditions, does not work. However, in these cases we can apply the method developed here.

If we start with Eq. (3.23), the drift and the noise vector fields are [cf. Eq. (2.21)]

$$F = -\left(a + \frac{b}{1+x_1^2+x_2^2}\right)\left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}\right) \tag{3.25}$$

$$G = \frac{1}{1+x_1^2+x_2^2}\left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}\right) \tag{3.26}$$

The first compatibility condition, Eq. (3.16), is obviously satisfied because the drift vector is proportional to the noise vector, their ratio being

$$\lambda(x_1, x_2) = -2[b + a(1+x_1^2+x_2^2)] \tag{3.27}$$

In this particular gross variable system the stationary distribution can be written as

$$\Pi_{\text{st}} = P_{\text{st}}(x_1, x_2)\, dx_1 \wedge dx_2 \tag{3.28}$$

which, introduced into Eq. (3.17) along with Eqs. (3.25) and (3.26), gives [cf. Eq. (3.10)]

$$x_1 \frac{\partial P_{\text{st}}}{\partial x_1} + x_2 \frac{\partial P_{\text{st}}}{\partial x_2} + \left[\frac{2}{1+x_1^2+x_2^2} - (1+x_1^2+x_2^2)\,\lambda(x_1, x_2)\right] P_{\text{st}} = 0 \tag{3.29}$$

whose general solution is

$$P_{\text{st}}(z) = N(\arg z)\,\frac{1+|z|^2}{|z|^{2(a+b+1)}} \exp\left[-|z|^2\left(2a+b+\frac{1}{2}a\,|z|^2\right)\right] \tag{3.30}$$

Doing an analogous calculation, we find the stationary distribution for the model represented by Eq. (3.22). The final result is

$$P_{\text{st}}(z) = N(\arg z)\,|z|^{\alpha-1} \exp\left(-\frac{\beta}{\gamma}|z|^{2\gamma}\right) \tag{3.31}$$

In these expressions $N(\arg z)$ is an arbitrary function of $\arg z$ to be determined by suitable boundary conditions and by the normalization of the stationary distribution.

4. CONCLUSION

From an intrinsic viewpoint, the dynamic evolution of a stochastic system that satisfies a Langevin equation is determined by the integral curves of an intrinsic field defined by

$$X = \partial/\partial t + F + \xi^k(t)\, G_k \tag{4.1}$$

where F and G_k are defined by Eqs. (2.2). Nevertheless, the final state is not determined, since the evolution does not take place following a simple trajectory, but rather by means of an ensemble of trajectories, each one corresponding to a concrete realization of the noise. We should point out that this formalism can also facilitate the systematic study of the invariants and symmetries of the stochastic process, sine a time-independent tensor $\mathbf{T}$ is invariant under the dynamic evolution of the system (i.e., invariant under a group of transformations generated by the field $X^{(19)}$) if, and only if, the following $m+1$ (partial differential) equations hold[19]:

$$L_F \mathbf{T} = 0; \qquad L_{G_k} \mathbf{T} = 0, \qquad k = 1,..., m \tag{4.2}$$

These equations allows us to know, in each concrete case, whether or not there exist invariants of the stochastic process and to determine them.

In the case of Gaussian white noise the intrinsic Langevin equation (4.1) is stochastically equivalent to the intrinsic Fokker–Planck equation:

$$\dot{\Pi} = -L_F \Pi + \frac{1}{2}\, \delta^{jk} L_{G_j} L_{G_k} \Pi \tag{4.3}$$

where $\Pi = P(\mathbf{q}, t)\, dq^1 \wedge \cdots \wedge dq^n$ is the n-form of probability. To arrive at this equation, we have not had to define any kind of metric tensor in the phase space.

Finally, we have also found that if the vector fields F and G_k satisfy the conditions:

(i) $$F = \frac{1}{2} \sum_{j=1}^{m} \lambda_j(q)\, G_j \tag{4.4}$$

(ii) The system

$$L_{G_\alpha} \Pi_{\text{st}} = \lambda_\alpha(\mathbf{q})\, \Pi_{\text{st}} \tag{4.5}$$

is complete (or it has been complete), then the solution Π_{st} of Eq. (4.5) is the stationary distribution of the Fokker–Planck equation (4.3) such that the associated probability current vanishes everywhere. This method generalizes the potential conditions, because it can be used even though the diffusion matrix is singular. In this case, the method provides a systematic way of finding the stationary distribution when, as often happens, it is not trivial to find it by inspection of the Fokker–Planck equation.

APPENDIX A

A1. Derivation of Eq. (2.13)

If f is a function, $X \in T(M)$, and $\alpha \in \Lambda(M)$, we have the following property of the Lie derivative[21]:

$$L_{fX}\alpha = fL_X\alpha + df \wedge i_X\alpha$$

In our case

$$L_{\xi^k G_k} = \xi^k L_{G_k}\Omega + d\xi^k \wedge i_{G_k}\Omega \tag{A.1}$$

Taking into account (2.10) and the fact that the interior product i_X is an antiderivation,[19] we have

$$i_{G_k}\Omega = i_{G_k}(N \wedge dt) = (i_{G_k}N) \wedge dt + (-1)^n N \wedge (i_{G_k} dt)$$

However, $d\xi^k = \dot{\xi}^k\, dt$; $i_{G_k} dt = 0$ (since G_k is independent of time); and $dt \wedge dt = 0$; therefore

$$d\xi^k \wedge i_{G_k}\Omega = 0 \tag{A.2}$$

On the other hand[19]

$$L_{G_k}(\xi^k\Omega) = \xi^k L_{G_k}\Omega + (G_k\xi^k)\,\Omega = \xi^k L_{G_k}\Omega \tag{A.3}$$

since $G_k\xi^k = 0$ because $\xi^k = \xi^k(t)$. Substitution of (A.3) and (A.2) into (A.1) gives Eq. (2.13).

A2. Derivation of Eq. (2.16)

If the noises $\xi^k(t)$ are Gaussian of zero mean, we may apply the Novikov theorem[28]:

$$\langle \xi^k(t)\,\Omega \rangle = \int_0^t dt'\, \langle \xi^k(t)\,\xi^j(t') \rangle \langle \delta\Omega(t)/\delta\xi^j(t') \rangle \tag{A.4}$$

(the symbol "δ" here denotes functional derivative). In the case of white noise

$$\langle \xi^k(t)\,\xi^j(t') \rangle = \delta^{jk}\,\delta(t-t') \tag{A.5}$$

Introducing (A.5) in (A.4) and applying the Stratonovich prescription over the integration of the Dirac function in the semi-interval,[29] we have

$$\langle \xi^k(t)\,\Omega \rangle = \frac{1}{2} \langle \delta\Omega/\delta\xi^k(t) \rangle \tag{A.6}$$

From Eqs. (3.8)–(2.10) we deduce

$$\frac{\delta\Omega(t)}{\delta\xi^j(t')} = -\frac{\partial}{\partial q^\mu} \left\{ \delta^n(\mathbf{q}-\boldsymbol{\phi}) \cdot \frac{\delta\phi^\mu(t; \mathbf{q}_0, t_0; [\xi])}{\delta\xi^j(t')} \right\} dq^1 \wedge \cdots \wedge dq^n \wedge dt$$

On the other hand, the formal solution of the Langevin equation (2.1) can be written in the form

$$\phi^\mu(t; \mathbf{q}_0, t_0; [\xi]) = q_0^\mu + \int_{t_0}^{t} dt\, [f^\mu(\boldsymbol{\phi}(s)) + \xi^k(s)\, g_k^\mu(\boldsymbol{\phi}(s))]$$

Taking the derivative of this equation gives $(0 < t' < t)$:

$$\frac{\delta\phi^\mu(t)}{\delta\xi^j(t')} = g_j^\mu(\boldsymbol{\phi}(t')) + \int_{t'}^{t} dt \frac{\partial}{\partial q^\nu} [f^\mu(\boldsymbol{\phi}(s)) + \xi^k(s)\, g_k^\mu(\boldsymbol{\phi}(s))] \frac{\delta\phi^\nu(s)}{\delta\xi^j(t')}$$

and therefore

$$\frac{\delta\phi^\mu(t)}{\delta\xi^j(t)} = g_j^\mu(\boldsymbol{\phi}(t))$$

whence

$$\frac{\delta\Omega(t)}{\delta\xi^j(t)} = -\frac{\partial}{\partial q^\mu} [g_j^\mu(\mathbf{q})\, \delta^n(\mathbf{q}-\boldsymbol{\phi})]\, dq^1 \wedge \cdots \wedge dq^n \wedge dt$$

and Eq. (A.6) becomes

$$\langle \xi^k(t)\,\Omega \rangle = -\frac{1}{2} \delta^{jk} \frac{\partial}{\partial q^\mu} [g_j^\mu(\mathbf{q}) \langle \delta^n(\mathbf{q}-\boldsymbol{\phi}) \rangle]\, dq^1 \wedge \cdots \wedge dq^n \wedge dt$$

but[19]

$$\frac{\partial}{\partial q^\mu} [g_k^\mu(q) \langle \delta^n(\mathbf{q}-\boldsymbol{\phi}) \rangle]\, dq^1 \wedge \cdots \wedge dq^n \wedge dt = L_{G_j} \langle \Omega \rangle$$

Therefore

$$\langle \xi^k(t)\,\Omega \rangle = -\frac{1}{2} \delta^{jk} L_{G_j} \langle \Omega \rangle$$

which is Eq. (2.16).

A3. Derivation of Eq. (2.19)

Considering[19,21]

$$L_X = di_x + i_X d$$

where d is the exterior derivative and i_X is the inner product associated to the field X, we have

$$L_{\partial/\partial t}\langle \Omega \rangle = (di_{\partial/\partial t} + i_{\partial/\partial t}\, d)(\Pi \wedge dt) = i_{\partial/\partial t}(d\Pi) \wedge dt = \dot{\Pi} \wedge dt$$

Equation (2.17) becomes

$$\left[\dot{\Pi} + L_F \Pi + \frac{1}{2}\, \delta^{jk}\, L_{G_k} L_{G_k} \Pi \right] \wedge dt = 0$$

That is

$$\dot{\Pi} + L_F \Pi + \frac{1}{2}\, \delta^{jk} L_{G_j} L_{G_k} \Pi = 0$$

since F and G_k are time-independent and Π does not contain dt.

A4. Derivation of Eq. (3.10)

From Eq. (2.18) we have

$$\begin{aligned} L_{G_k} \Pi_{\mathrm{st}} &= L_{G_k}[P_{\mathrm{st}}(\mathbf{q})\, dq^1 \wedge \cdots \wedge dq^n] \\ &= [L_{G_k} P_{\mathrm{st}}(\mathbf{q})]\, dq^1 \wedge \cdots \wedge dq^n + P_{\mathrm{st}}(\mathbf{q}) \\ &\quad \times \left[\sum_{\alpha=1}^{n} dq^1 \wedge \cdots \wedge (L_{G_k}\, dq^\alpha) \wedge \cdots \wedge dq^n \right] \end{aligned}$$

(here we *do not* sum over repeated indices). But[19]

$$L_{G_k} P_{\mathrm{st}}(\mathbf{q}) = G_k(P_{\mathrm{st}}(\mathbf{q})) = \sum_{\alpha=1}^{n} g_k^\alpha(\mathbf{q})\, \frac{\partial P_{\mathrm{st}}(\mathbf{q})}{\partial q^\alpha}$$

and

$$\begin{aligned} dq^1 &\wedge \cdots \wedge (L_{G_k}\, dq^\alpha) \wedge \cdots \wedge dq^n \\ &= dq^1 \wedge \cdots \wedge \left(\sum_{\beta=1}^{n} \frac{\partial g_k^\alpha}{\partial q^\beta}\, dq^\beta \right) \wedge \cdots \wedge dq^n \\ &= \frac{\partial g_k^\alpha}{\partial q^\alpha}\, dq^1 \wedge \cdots \wedge dq^\alpha \wedge \cdots \wedge dq^n \end{aligned}$$

(since $dq^\beta \wedge dq^\beta = 0$). Thus

$$L_{G_k} \Pi_{\text{st}} = \left[\sum_{\alpha=1}^{n} \left(g_k^\alpha \frac{\partial P_{\text{st}}}{\partial q^\alpha} \right) + P_{\text{st}} \left(\sum_{\alpha=1}^{n} \frac{\partial g_k^\alpha}{\partial q^\alpha} \right) \right] dq^1 \wedge \cdots \wedge dq^n$$

and Eq. (3.9) becomes

$$\sum_{\alpha=1}^{n} \left(g_k^\alpha \frac{\partial P_{\text{st}}}{\partial q^\alpha} \right) + P_{\text{st}} \left(\sum_{\alpha=1}^{n} \frac{\partial g_k^\alpha}{\partial q^\alpha} \right) = \lambda_k P_{\text{st}}, \qquad k = 1,..., m$$

which is Eq. (3.10).

APPENDIX B. COMPLETION OF THE SYSTEM (3.9)

If the set of noise vector fields $\{G_i;\, i = 1,..., m\}$ is not closed under the Lie bracket, we may proceed as follows:

1. We first look for the minimal extension of the set of vector fields $\{G_i;\, i = 1,..., m\}$ such that the final set is closed under the Lie bracket. This extended set will be denoted by

$$\{\tilde{G}_\alpha; \quad \alpha = 1,..., r\}, \qquad n \geqslant r \geqslant m$$

under the conditions that (a) $\tilde{G}_i = G_i$, $i = 1,..., m$, and (b) there exist functions $\tilde{C}^\gamma_{\alpha\beta}$ such that

$$[\tilde{G}_\alpha, \tilde{G}_\beta] = \tilde{C}^\gamma_{\alpha\beta} \tilde{G}_\gamma; \qquad \alpha, \beta, \gamma = 1,..., r$$

The latter extension can be obtained by supplementing the primary set $\{G_i,\, i = 1,..., m\}$ by those Lie brackets $[G_i, G_j]$, $i, j = 1,..., m$, that do not depend linearly on $G_1,..., G_m$.

If after this first extension the resulting set $G_1,..., G_m, \tilde{G}_{m+1},..., \tilde{G}_{s_1}$ was not closed under the Lie bracket, we would repeat the process as many times as necessary.[23] Note that since the phase space is finite-dimensional (n), this interaction process necessarily ends with $r \leqslant n$.

2. The complete set corresponding to (3.9) has the final form

$$L_{\tilde{G}_\alpha} \Pi_{\text{st}} = \bar{\lambda}_\alpha(\mathbf{q})\, \Pi_{\text{st}}; \qquad \alpha = 1,..., r$$

with $\tilde{G}_i = G_i$ and $\bar{\lambda}_i(\mathbf{q}) = \lambda_i(\mathbf{q})$ for $i = 1,..., m$; and since the remaining vector fields $\alpha = m + 1,..., r$ have the form

$$\tilde{G}_{\alpha_1} = [G_i, G_j]$$
$$\tilde{G}_{\alpha_2} = [G_i, [G_j, G_k]]; \qquad i, j, k = \lambda_1,..., m; \qquad \alpha_1, \alpha_2 = m + 1,..., r$$

and so on, the corresponding functions $\tilde{\lambda}_\alpha(\mathbf{q})$ $(\alpha = m+1,..., r)$ must therefore be taken such that

$$\tilde{\lambda}_{\alpha_1} \equiv G_i \lambda_j - G_j \lambda_i$$
$$\tilde{\lambda}_{\alpha_2} \equiv G_i G_j \lambda_k - G_i G_k \lambda_j - G_j G_k \lambda_i + G_k G_j \lambda_i$$

and so on.

REFERENCES

1. A. Kolmogoroff, *Math. Ann.* **113**:766 (1937).
2. A. M. Yoglom, *Mat. Sb.* **24**:457 (1949).
3. R. L. Stratonovich, *J. SIAM Control* **4**:362 (1966).
4. R. Graham, *Z. Phys. B* **26**:397 (1977).
5. L. Garrido, *Physica* **100A**:140 (1980).
6. H. Grabert, R. Graham, and M. S. Green, *Phys. Rev. A* **21**:2136 (1980).
7. L. Garrido and J. Masoliver, *J. Math. Phys.* **23**:1155 (1982).
8. J. Masoliver, L. Garrido, and J. Llosa, *Phys. Lett.* **102A**:93 (1984).
9. J. Masoliver and L. Garrido, *Phys. Lett.* **103*A***:366 (1984).
10. K. D. Elworthy, *Stochastic Differential Equations on Manifolds* (Cambridge University Press, 1982).
11. U. R. Steiger, *Physica* **125A**:1 (1984).
12. M. van den Berg and J. T. Lewis, *Bull. Lond. Math. Soc.* **17**:144 (1985).
13. L. Garrido and J. Llosa, in *Systems Far from Equilibrium*, L. Garrido, ed. (Springer-Verlag, 1980).
14. C. W. Gardiner, *Handbook of Stochastic Methods* (Springer-Verlag, 1983).
15. V. I. Klyatskin and V. E. Tatarski, *Sov. Phys. Usp.* **16**:494 (1973).
16. P. Hänggi, *Z. Phys. B* **31**:407 (1978).
17. N. G. Van Kampen, *Phys. Rep.* **24**:171 (1976).
18. H. J. Sussmann, *Ann. Prob.* **6**, 19–41 (1978).
19. Y. Choquet-Bruhat, C. DeWitt-Morette, and M. Dillard-Bleick, *Analysis, Manifolds and Physics* (North-Holland, 1977).
20. H. Flanders, *Differential Forms* (Academic Press, 1963).
21. C. Godbillon, *Géométrie Différentielle et Mécanique Analitique* (Hermann, Paris, 1969).
22. R. L. Stratonovich, *Topics in the Theory of Random Noise*, Vol. I (Gordon and Breach, 1963).
23. E. Goursat, *A Course in Mathematical Analysis*, Vol. II, Part II (Dover, 1959).
24. S. L. McCall, *Phys. Rev. A* **9**:1515 (1974).
25. H. Haken, *Laser Theory* (Springer-Verlag, 1984).
26. A. Schenzle and H. Brand, *Phys. Rev. A* **20**:1628 (1979).
27. R. Graham and A. Schenzle, *Phys. Rev. A* **23**:1302 (1981).
28. E. A. Novikov, *Sov. Phys. JETP* **20**:1290 (1965).
29. R. L. Stratonovich, *Conditional Markov Processes and Their Application to the Theory of the Optimal Control* (Elsevier, 1968).